LOWER BOUNDS FOR MOMENTS OF L-FUNCTIONS: SYMPLECTIC AND ORTHOGONAL EXAMPLES

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1. Introduction

An important problem in number theory asks for asmptotic formulas for the moments of central values of L-functions varying in a family. This problem has been intensively studied in recent years, and thanks to the pioneering work of Keating and Snaith [7], and the subsequent contributions of Conrey, Farmer, Keating, Rubinstein and Snaith [1], and Diaconu, Goldfeld and Hoffstein [3] there are now well-established conjectures for these moments. The conjectured asymptotic formulas take different shapes depending on the symmetry group attached to the family of L-functions, given in the work of Katz and Sarnak [6], with three classes of formulas depending on whether the group in question is unitary, orthogonal or symplectic. While there are many known examples of asymptotic formulas dealing with the first few moments of a family of L-functions, in general the moment conjectures seem formidable. In [8] we recently gave a simple method to obtain lower bounds of the conjectured order of magnitude in many families of L-functions. In [8] we illustrated our method by working out lower bounds for $\sum^{*}_{\chi \pmod q} |L(\frac{1}{2}, \chi)|^{2k}$ where q is a large prime, and the sum is over the primitive Dirichlet L-functions $\pmod q$. This was an example of a 'unitary' family of L-functions, and in this paper we round out the picture by providing lower bounds for moments of L-functions arising from orthogonal and symplectic families.

As our first example, we consider $\mathcal{H}_k$ the set of Hecke eigencuspforms of weight k for the full modular group $SL(2, \mathbb{Z})$. We will think of the weight k as being large, and note that $\mathcal{H}_k$ contains about $k/12$ forms. Given $f \in \mathcal{H}_k$ we write its Fourier expansion as

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz),$$

where we have normalized the Fourier coefficients so that the Hecke eigenvalues $\lambda_f(n)$ satisfy Deligne's bound $|\lambda_f(n)| \leq \tau(n)$ where $\tau(n)$ is the number of divisors of n. Consider the associated L-function

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s} = \prod_{p} (1 - \lambda_f(p) p^{-s} + p^{-2s})^{-1},$$

which converges absolutely in Re $(s) > 1$ and extends analytically to the entire complex plane. Recall that $L(s, f)$ satisfies the functional equation

$$\Lambda(s, f) := (2\pi)^{-s} \Gamma(s + \tfrac{k-1}{2}) L(s, f) = i^k \Lambda(1 - s, f).$$

If $k \equiv 2 \pmod 4$ then the sign of the functional equation is negative and so $L(\tfrac{1}{2}, f) = 0$. We will therefore assume that $k \equiv 0 \pmod 4$.

While dealing with moments of L-functions in $\mathcal{H}_k$, it is convenient to use the natural 'harmonic weights' that arise from the Petersson norm of f. Define the weight

$$\omega_f := \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \langle f, f \rangle = \frac{k-1}{2\pi^2} L(1, \mathrm{Sym}^2 f),$$

where $\langle f, f \rangle$ denotes the Petersson inner product. For a typical f in $\mathcal{H}_k$ the harmonic weight ω_f is of size about $k/12$, and so $\sum_{f \in \mathcal{H}_k} \omega_f^{-1}$ is very nearly 1. The weights ω_f arise naturally in connection with the Petersson formula, and the facts mentioned above are standard and may be found in Iwaniec [4].

For a positive integer r, we are interested in the r-th moment

$$\sum_{f \in \mathcal{H}_k}^{h} L(\tfrac{1}{2}, f)^r := \sum_{f \in \mathcal{H}_k} \frac{1}{\omega_f} L(\tfrac{1}{2}, f)^r.$$

This family of L-functions is expected to be of 'orthogonal type' and the Keating-Snaith conjectures predict that for any given $r \in \mathbb{N}$ as $k \to \infty$ with $k \equiv 0 \pmod 4$ we have

$$\sum_{f \in \mathcal{H}_k}^{h} L(\tfrac{1}{2}, f)^r \sim C(r)(\log k)^{r(r-1)/2},$$

for some positive constant $C(r)$. This conjecture can be verified for $r = 1$ and $r = 2$, and if we permit an additional averaging over the weight k then for $r = 3$ and 4 also.

Theorem 1. *For any given even natural number r, and weight $k \geq 12$ with $k \equiv 0 \pmod 4$, we have*

$$\sum_{f \in \mathcal{H}_k}^{h} L(\tfrac{1}{2}, f)^r \gg_r (\log k)^{r(r-1)/2}.$$

In fact, with more effort our method could be adapted to give lower bounds as in Theorem 1 for all rational numbers $r \geq 1$, rather than just even integers.

Our other example involves the family of quadratic Dirichlet L-functions. Let d denote a fundamental discriminant, and let χ_d denote the corresponding real primitive character with conductor $|d|$. We are interested in the class of quadratic Dirichlet L-functions $L(s, \chi_d)$. Recall that, with $\mathfrak{a} = 0$ or 1 depending on whether d is positive or negative, these L-functions satisfy the functional equation

$$\Lambda(s, \chi_d) := \left(\frac{q}{\pi}\right)^{\frac{s+\mathfrak{a}}{2}} \Gamma\left(\frac{s+\mathfrak{a}}{2}\right) L(s, \chi_d) = \Lambda(1 - s, \chi_d).$$

Notice that the sign of the functional equation is always positive, and it is expected that the central values $L(\frac{1}{2}, \chi_d)$ are all positive although this remains unknown. This family is expected to be of 'symplectic' type and the Keating-Snaith conjectures predict that for any given $k \in \mathbb{N}$ and as $X \to \infty$ we have

$$\sum_{|d| \leq X}^{\flat} L(\tfrac{1}{2}, \chi_d)^k \sim D(k) X (\log X)^{k(k+1)/2},$$

for some positive constant $D(k)$, where the $\flat$ indicates that the sum is over fundamental discriminants. Jutila [5] established asymptotics for the first two moments of this family, and the third moment was evaluated in Soundararajan [10].

Theorem 2. *For every even natural number k we have*

$$\sum_{|d| \leq X}^{\flat} L(\tfrac{1}{2}, \chi_d)^k \gg_k X (\log X)^{k(k+1)/2}.$$

As with Theorem 1, our method can be used to obtain lower bounds for these moments for all rational numbers k, taking care to replace $L(\frac{1}{2}, \chi_d)^k$ by $|L(\frac{1}{2}, \chi_d)|^k$ when k is not an even integer. In the case of the fourth moment we are able to get a lower bound $\geq (D(4) + o(1)) X (\log X)^{10}$, which matches exactly the asymptotic conjectured by Keating and Snaith. The details of this calculation will appear elsewhere.

2. Proof of Theorem 1

Let $x := k^{\frac{1}{2r}}$ and consider

$$A(f) := A(f, x) = \sum_{n \leq x} \frac{\lambda_f(n)}{\sqrt{n}}.$$

We will consider

$$S_1 := \sum_{f \in \mathcal{H}_k}^h L(\tfrac{1}{2}, f) A(f)^{r-1}, \quad \text{and} \quad S_2 := \sum_{f \in \mathcal{H}_k}^h A(f)^r.$$

Then Hölder's inequality gives, keeping in mind that r is even so that $|A(f)|^r = A(f)^r$,

$$\left(\sum_{f \in \mathcal{H}_k}^h L(\tfrac{1}{2}, f) A(f)^{r-1} \right)^r \leq \left(\sum_{f \in \mathcal{H}_k}^h L(\tfrac{1}{2}, f)^r \right) \left(\sum_{f \in \mathcal{H}_k}^h A(f)^r \right)^{r-1},$$

so that

$$\sum_{f \in \mathcal{H}_k}^h L(\tfrac{1}{2}, f)^r \geq \frac{S_1^r}{S_2^{r-1}}.$$

We will prove Theorem 1 by finding the asymptotic orders of magnitude of S_1 and S_2.

We begin with S_2. To evaluate this, we must expand out $A(f)^r$ and group terms using the Hecke relations. To do this conveniently, let us denote by $\mathcal{H}$ the ring generated over the integers by symbols $x(n)$ ($n \in \mathbb{N}$) subject to the Hecke relations

$$x(1) = 1, \qquad \text{and} \qquad x(m)x(n) = \sum_{d | (m,n)} x\left(\frac{mn}{d^2}\right).$$

Thus $\mathcal{H}$ is a polynomial ring on $x(p)$ where p runs over all primes. Using the Hecke relations we may write

$$x(n_1) \cdots x(n_r) = \sum_{t | \prod_{j=1}^r n_j} b_t(n_1, \ldots, n_r) x(t),$$

for certain integers $b_t(n_1, \ldots, n_r)$. Note that $b_t(n_1, \ldots, n_r)$ is symmetric in the variables $n_1, \ldots, n_r$, and that $b_t(n_1, \ldots, n_r)$ is always non-negative, and finally that $b_t(n_1, \ldots, n_r) \leq \tau(n_1) \cdots \tau(n_r) \ll (n_1 \cdots n_r)^\epsilon$. Of special importance for us will be the coefficient of $x(1)$ namely $b_1(n_1, \ldots, n_r)$. It is easy to see that b_1 satisfies a multiplicative property: if $(\prod_{j=1}^r m_j, \prod_{j=1}^r n_j) = 1$ then

$$b_1(m_1 n_1, \ldots, m_r n_r) = b_1(m_1, \ldots, m_r) b_1(n_1, \ldots, n_r).$$

Thus it suffices to understand b_1 when the $n_1, \ldots, n_r$ are all powers of some prime p. Here we note that $b_1(p^{a_1}, \ldots, p^{a_r})$ is independent of p, always lies between 0 and $(1 + a_1) \cdots (1 + a_r)$, and that it equals 0 if $a_1 + \ldots + a_r$ is odd. If we write

$$B_r(n) = \sum_{\substack{n_1, \ldots, n_r \\ n_1 \cdots n_r = n}} b_1(n_1, \ldots, n_r),$$

then we find that $B_r(n)$ is a multiplicative function, that $B_r(n) = 0$ unless n is a square, and that $B_r(p^a)$ is independent of p and grows at most polynomially in a. Finally, and crucially, we note that

$$B_r(p^2) = r(r-1)/2,$$

which follows upon noting that $b_1(p^2, 1, \ldots, 1) = 0$ and that $b_1(p, p, 1, \ldots, 1) = 1$.

Returning to S_2 note that

$$A(f)^r = \sum_{n_1, \ldots, n_r \leq x} \frac{1}{\sqrt{n_1 \cdots n_r}} \sum_{t | n_1 \cdots n_r} b_t(n_1, \ldots, n_r) \lambda_f(t),$$

and so we require knowledge of $\sum_{f \in \mathcal{H}_k}^h \lambda_f(t)$. This follows easily from Petersson's formula.

Lemma 2.1. *If k is large, and t and u are natural numbers with $tu \leq k^2/10^4$ then*

$$\sum_{f \in \mathcal{H}_k}^h \lambda_f(t) \lambda_f(u) = \delta(t, u) + O(e^{-k}),$$

where $\delta(t, u)$ is 1 if $t = u$ and is 0 otherwise.

Proof. Petersson's formula (see [4]) gives

$$\sum_{f \in \mathcal{H}_k}^h \lambda_f(t)\lambda_f(u) = \delta(t, u) + 2\pi i^k \sum_{c=1}^{\infty} \frac{S(t, u; c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{tu}}{c}\Big).$$

Note that if $z \le 2k$ then $(z/2)^{k-1+\ell}/\Gamma(k-1+\ell) \le (z/2)^{k-1}/\Gamma(k-1)$ for all $\ell \ge 0$. We now use the series representation for $J_{k-1}(z)$ which gives, for $z \le 2k$,

$$|J_{k-1}(z)| = \Big| \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (z/2)^\ell}{\ell!} \frac{(z/2)^{\ell+k-1}}{\Gamma(\ell+k-1)} \Big| \le \frac{(z/2)^{k-1}}{\Gamma(k-1)} e^{z/2}.$$

Therefore, for $tu \le k^2/10^4$, we deduce that

$$\Big| J_{k-1}\Big(\frac{4\pi\sqrt{tu}}{c}\Big) \Big| \le \frac{(2\pi k/(100c))^{k-1}}{(k-2)!} e^{\pi k/50}.$$

Using the trivial bound $|S(t, u; c)| \le c$ we conclude that

$$2\pi i^k \sum_{c=1}^{\infty} \frac{S(t, u; c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{tu}}{c}\Big) \ll \Big(\frac{\pi k}{50}\Big)^{k-1} \frac{1}{(k-2)!} e^{\pi k/50} \ll e^{-k},$$

for large k, as desired.

Since $n_1 \cdots n_r \le x^r = \sqrt{k}$ we see by Lemma 2.1 that

$$S_2 = \sum_{n_1, \ldots, n_r \le x} \frac{b_1(n_1, \ldots, n_r)}{\sqrt{n_1 \cdots n_r}} + O\Big(e^{-k} \sum_{n_1, \ldots n_r \le x} \frac{\tau(n_1) \cdots \tau(n_r)\tau(n_1 \cdots n_r)}{\sqrt{n_1 \cdots n_r}} \Big).$$

The error term is easily seen to be $\ll e^{-k} x^k = k^{\frac{1}{2}} e^{-k}$, a negligible amount. As for the main term we see easily that

$$\sum_{n \le x} \frac{B_r(n)}{\sqrt{n}} \le \sum_{n_1, \ldots, n_r \le x} \frac{b_1(n_1, \ldots, n_r)}{\sqrt{n_1 \cdots n_r}} \le \sum_{n \le x^r} \frac{B_r(n)}{\sqrt{n}}.$$

Recall that $B_r(n)$ is a multiplicative function with $B_r(p) = 0$, $B_r(p^2) = r(r-1)/2$ and $B_r(p^a)$ grows only polynomially in a. Thus the generating function $\sum_{n=1}^{\infty} B_r(n)n^{-s}$ can be compared with $\zeta(2s)^{r(r-1)/2}$, the quotient being a Dirichlet series absolutely convergent in $\mathrm{Re}(s) > \frac{1}{4}$. A standard argument (see Theorem 2 of [9]) therefore gives that

$$\sum_{n \le z} \frac{B_r(n)}{\sqrt{n}} \sim C_r(\log z)^{r(r-1)/2},$$

for a positive constant C_r. It follows that

$$S_2 \asymp (\log x)^{r(r-1)/2} \asymp (\log k)^{r(r-1)/2}.$$

We now turn to S_1. To evaluate S_1 we need an 'approximate functional equation' for $L(\frac{1}{2}, f)$.

Lemma 2.2. *Define for any positive number ξ the weight*

$$W_k(\xi) := \frac{1}{2\pi i} \int_{(c)} \frac{(2\pi)^{-\frac{1}{2}-s}\Gamma(s+\frac{k}{2})}{(2\pi)^{-\frac{1}{2}}\Gamma(\frac{k}{2})}\xi^{-s}\frac{ds}{s},$$

where the integral is over a vertical line $c - i\infty$ to $c + i\infty$ with $c > 0$. Then, for $k \equiv 0$ (mod 4),

$$L(\tfrac{1}{2}, f) = 2\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{\sqrt{n}}W_k(n) = 2\sum_{n\leq k} \frac{\lambda_f(n)}{\sqrt{n}}W_k(n) + O(e^{-k}).$$

Further the weight $W_k(\xi)$ satisfies $|W_k(\xi)| \ll k\pi^{-k}/\xi$ for $\xi > k$, $W_k(\xi) = 1 + O(e^{-k})$ for $\xi < k/100$, and $W_k(\xi) \ll 1$ for $k/100 \leq \xi \leq k$.

Proof. The argument is standard. For $1 \leq c > \frac{1}{2}$ we consider

$$I = \frac{1}{2\pi i} \int_{(c)} \frac{(2\pi)^{-\frac{1}{2}-s}\Gamma(s+\frac{k}{2})}{(2\pi)^{-\frac{1}{2}}\Gamma(\frac{k}{2})}L(\tfrac{1}{2} + s, f)\frac{ds}{s}.$$

Expanding out $L(\frac{1}{2} + s, f)$ and integrating term by term we see that

$$I = \sum_{n=1}^{\infty} \lambda_f(n)n^{-\frac{1}{2}}W_k(n).$$

On the other hand moving the line of integration to the line $\mathrm{Re}(s) = -c$ we see that

$$I = L(\tfrac{1}{2}, f) + \frac{1}{2\pi i} \int_{(-c)} \frac{\Lambda(\tfrac{1}{2} + s, f)}{(2\pi)^{-\frac{1}{2}}\Gamma(\frac{k}{2})}\frac{ds}{s},$$

and using the functional equation $\Lambda(\frac{1}{2} + s, f) = \Lambda(\frac{1}{2} - s, f)$, and replacing s by $-s$ in the integral above, we see that $I = L(\frac{1}{2}, f) - I$. Thus

$$L(\tfrac{1}{2}, f) = 2I = 2\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{\sqrt{n}}W_k(n).$$

Regarding the weight $W_k(\xi)$ note that by considering the integral for some large positive integer c we get that

$$|W_k(\xi)| \leq \frac{1}{2\pi} \int_{(c)} (2\pi\xi)^{-c}\frac{|\Gamma(s+\frac{k}{2}+1)|}{\Gamma(\frac{k}{2})}\frac{|ds|}{|s(s+k/2)|}$$

$$\leq (2\pi\xi)^{-c}\frac{\Gamma(c+1+\frac{k}{2})}{\Gamma(\frac{k}{2})} \leq (2\pi\xi)^{-c}(k+c)^c.$$

Taking $c = k$ we obtain that $|W_k(\xi)| \leq (k/(\pi\xi))^k$ so that if $\xi \geq k$ then $|W_k(\xi)| \leq (k/\xi)\pi^{-(k-1)}$. This proves the first bound for $W_k(\xi)$ claimed in the Lemma, and also shows that

$$\left| \sum_{n>k} \frac{\lambda_f(n)}{\sqrt{n}} W_k(n) \right| \ll k\pi^{-k} \sum_{n>k} \frac{|\lambda_f(n)|}{n^{\frac{3}{2}}} \ll e^{-k}.$$

The other claims on $W_k(\xi)$ are proved similarly; for the range $\xi < k/100$ we move the line of integration to $c = -\frac{k}{2} + 1$, for the last range $k/100 \leq \xi \leq k$ just take the integral to be on the line $c = 1$.

Returning to S_1 note that

$$A(f)^{r-1} = \sum_{n_1,\dots,n_{r-1} \leq x} \frac{1}{\sqrt{n_1 \cdots n_{r-1}}} \sum_{t | n_1 \cdots n_{r-1}} b_t(n_1,\dots,n_{r-1})\lambda_f(t).$$

Since $A(f)^{r-1}$ is trivially seen to be $\ll x^{r-1} < \sqrt{k}$, we see by Lemma 2.2, that

$$S_1 = 2\sum_{n \leq k} \frac{1}{\sqrt{n}} W_k(n) \sum_{n_1,\dots,n_{r-1} \leq x} \sum_{t | n_1 \cdots n_{r-1}} \frac{b_t(n_1,\dots,n_{r-1})}{\sqrt{n_1 \cdots n_{r-1}}} \sum_{f \in \mathcal{H}_k}^h \lambda_f(t)\lambda_f(n) + O(\sqrt{k}e^{-k}).$$

Now we appeal to Lemma 2.1. The error term that arises is trivially bounded by $\ll ke^{-k}$ which is negligible. In the main term $\delta(n,t)$, since $t \leq x^{r-1} < \sqrt{k}$ we may replace $W_k(n)$ by $1 + O(e^{-k})$. It follows that

$$S_1 = 2 \sum_{n_1,\dots,n_{r-1} \leq x} \sum_{t | n_1 \cdots n_{r-1}} \frac{b_t(n_1,\dots,n_{r-1})}{\sqrt{n_1 \cdots n_{r-1}}} \frac{1}{\sqrt{t}} + O(ke^{-k}).$$

Now observe that $b_1(n_1,\dots,n_{r-1},t) = b_t(n_1,\dots,n_{r-1})$ if t divides $n_1 \cdots n_{r-1}$, and otherwise $b_1(n_1,\dots,n_{r-1},t)$ is zero. Therefore, writing n_r for t, we obtain that

$$S_1 = 2 \sum_{n_1,\dots,n_{r-1} \leq x} \sum_{n_r \leq \sqrt{k}} \frac{b_1(n_1,\dots,n_r)}{\sqrt{n_1 \cdots n_r}} + O(ke^{-k}).$$

Using $b_1 \geq 0$ we see that $S_1 \geq 2S_2 + O(ke^{-k})$, and moreover, arguing as in the case of S_2 we may see that

$$S_1 \asymp (\log k)^{r(r-1)/2}.$$

Theorem 1 follows.

3. Proof of Theorem 2

For simplicity, we will restrict ourselves to fundamental discriminants of the form $8d$ where d is a positive, odd square-free number with $X/16 < d \leq X/8$. Let k be a given even number, and set $x = X^{\frac{1}{10k}}$. Define

$$A(8d) := \sum_{n \leq x} \frac{\chi_{8d}(n)}{\sqrt{n}},$$

and let

$$S_1 := \sum_{X/16 < d \leq X/8} \mu^2(2d) L(\tfrac{1}{2}, \chi_{8d}) A(8d)^{k-1}, \quad \text{and} \quad S_2 := \sum_{X/16 < d \leq X/8} \mu^2(2d) A(8d)^k.$$

An application of Hölder's inequality gives that

$$\sideset{}{^\flat}\sum_{|d| \leq X} L(\tfrac{1}{2}, \chi_{8d})^k \geq \sum_{X/16 < d \leq X/8} \mu^2(2d) L(\tfrac{1}{2}, \chi_{8d})^k \geq \frac{S_1^k}{S_2^{k-1}},$$

so that to prove Theorem 2 we need only give satisfactory estimates for S_1 and S_2.

We start with S_2. Expanding our $A(8d)^k$ we see that

$$(3.1) \qquad S_2 = \sum_{n_1, \dots, n_k \leq x} \frac{1}{\sqrt{n_1 \cdots n_k}} \sum_{X/16 < d \leq X/8} \mu^2(2d) \left(\frac{8d}{n_1 \cdots n_k} \right).$$

Lemma 3.1. *Let n be an odd integer, and let $z \geq 3$ be a real number. If n is not a perfect square then*

$$\sum_{d \leq z} \mu^2(2d) \left(\frac{8d}{n} \right) \ll z^{\frac{1}{2}} n^{\frac{1}{4}} \log(2n),$$

while if n is a perfect square then

$$\sum_{d \leq z} \mu^2(2d) \left(\frac{8d}{n} \right) = \frac{z}{\zeta(2)} \prod_{p | 2n} \left(\frac{p}{p+1} \right) + O(z^{\frac{1}{2}+\epsilon} n^\epsilon).$$

Proof. Note that $\sum_{\alpha^2 | d} \mu(\alpha) = 1$ if d is square-free and 0 otherwise. Therefore,

$$\sum_{n \leq z} \mu^2(2d) \left(\frac{8d}{n} \right) = \sum_{\substack{\alpha \leq \sqrt{z} \\ \alpha \text{ odd}}} \left(\frac{8\alpha^2}{n} \right) \sum_{\substack{d \leq z/\alpha^2 \\ d \text{ odd}}} \left(\frac{d}{n} \right).$$

If n is not a square then the inner sum over d is a character sum to a non-principal character of modulus $2n$ (we take $2n$ to account for d being odd), and the Pólya-Vinogradov inequality (see [2]) gives that the sum over d is $\ll \sqrt{n} \log(2n)$. Further, the sum over d is trivially $\ll z/\alpha^2$. Thus, if n is not a square, we get that

$$\sum_{n \leq z} \mu^2(2d) \left(\frac{8d}{n} \right) \ll \sum_{\alpha \leq \sqrt{z}} \min \left(\sqrt{n} \log(2n), \frac{z}{\alpha^2} \right) \ll z^{\frac{1}{2}} n^{\frac{1}{4}} \log(2n),$$

upon using the first bound for $\alpha \leq z^{\frac{1}{2}} n^{-\frac{1}{4}}$ and the second bound for larger α.

If n is a perfect square, then $\left(\frac{8d}{n} \right) = 1$ if d is coprime to n, and is 0 otherwise. Thus

$$\sum_{d \leq z} \mu(2d)^2 \left(\frac{8d}{n} \right) = \sum_{\substack{d \leq z \\ (d, 2n) = 1}} \mu^2(d) = \frac{z}{\zeta(2)} \prod_{p | 2n} \left(\frac{p}{p+1} \right) + O(z^{\frac{1}{2}+\epsilon} n^\epsilon),$$

by a standard argument.

Using Lemma 3.1 in (3.1) we obtain that

$$S_2 = \frac{X}{16\zeta(2)} \sum_{\substack{n_1,\ldots,n_k \leq x \\ n_1\cdots n_k = \text{ odd square}}} \frac{1}{\sqrt{n_1\cdots n_k}} \prod_{p|2n_1\cdots n_k} \left(\frac{p}{p+1}\right) + O\left(X^{\frac{1}{2}+\epsilon} x^{k(\frac{3}{4}+\epsilon)}\right).$$

Since $x = X^{\frac{1}{10k}}$ the error term above is $\ll X^{\frac{3}{5}}$. Writing $n_1\cdots n_k = m^2$ we see that

$$\sum_{\substack{m^2 \leq x \\ m \text{ odd}}} \frac{d_k(m^2)}{m} \prod_{p|2m} \left(\frac{p}{p+1}\right) \leq \sum_{\substack{n_1,\ldots,n_k \leq x \\ n_1\cdots n_k = \text{ odd square}}} \frac{1}{\sqrt{n_1\cdots n_k}} \prod_{p|2n_1\cdots n_k} \left(\frac{p}{p+1}\right)$$

$$\leq \sum_{\substack{m^2 \leq x^k \\ m \text{ odd}}} \frac{d_k(m^2)}{m} \prod_{p|2m} \left(\frac{p}{p+1}\right).$$

A standard argument (see Theorem 2 of [9]) shows that

$$\sum_{\substack{m \leq z \\ m \text{ odd}}} \frac{d_k(m^2)}{m} \prod_{p|2m} \left(\frac{p}{p+1}\right) \sim C(k)(\log z)^{k(k+1)/2},$$

for a positive constant $C(k)$. We conclude that

$$(3.2) \qquad\qquad S_2 \asymp X(\log X)^{k(k+1)/2}.$$

It remains to evaluate S_1. As before, we need an 'approximate functional equation' for $L(\frac{1}{2}, \chi_{8d})$.

Lemma 3.2. *For a positive number ξ define the weight*

$$W(\xi) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \xi^{-s} \frac{ds}{s},$$

where the integral is over a vertical line $c - i\infty$ to $c + i\infty$ with $c > 0$. Then, for any odd, positive, square-free number d we have

$$L(\tfrac{1}{2}, \chi_{8d}) = 2 \sum_{n=1}^{\infty} \frac{\chi_{8d}(n)}{\sqrt{n}} W\left(\frac{n\sqrt{\pi}}{\sqrt{8d}}\right).$$

The weight $W(\xi)$ is smooth and satisfies $W(\xi) = 1 + O(\xi^{\frac{1}{2}-\epsilon})$ for ξ small, and for large ξ satisfies $W(\xi) \ll e^{-\xi}$. Moreover the derivative $W'(\xi)$ satisfies $W'(\xi) \ll \xi^{\frac{1}{2}-\epsilon} e^{-\xi}$.

Proof. This is given in Lemmas 2.1 and 2.2 of [10], but for completeness we give a sketch. For some $1 \geq c > \frac{1}{2}$, we consider

$$\frac{1}{2\pi i} \int_{c)} \frac{(8d/\pi)^{\frac{s}{2}+\frac{1}{4}} \Gamma(\frac{s}{2} + \frac{1}{4})}{(8d/\pi)^{\frac{1}{4}} \Gamma(\frac{1}{4})} L(\tfrac{1}{2} + s, \chi_{8d}) \frac{ds}{s},$$

and argue exactly as in Lemma 2.2. This gives the desired formula for $L(\frac{1}{2}, \chi_{8d})$. The results on the weight $W(\xi)$ follow upon moving the line of integration to $\mathrm{Re}(s) = -\frac{1}{2} + \epsilon$ when ξ is small, and taking c to be an appropriately large positive number if ξ is large.

By Lemma 3.2 we see that
(3.3)
$$S_1 = 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{n_1,\ldots,n_{k-1} \leq x} \frac{1}{\sqrt{n_1 \cdots n_{k-1}}} \sum_{X/16 < d \leq X/8} \mu^2(2d) \left(\frac{8d}{nn_1 \cdots n_{k-1}} \right) W \left(\frac{n\sqrt{\pi}}{\sqrt{8d}} \right).$$

If $nn_1 \cdots n_{k-1}$ is not a square, then using Lemma 3.1 and partial summation we may see that
$$\sum_{X/16 < d \leq X/8} \mu^2(2d) \left(\frac{8d}{nn_1 \cdots n_{k-1}} \right) W \left(\frac{n\sqrt{\pi}}{\sqrt{8d}} \right) \ll X^{\frac{1}{2}} (nn_1 \cdots n_{k-1})^{\frac{1}{4}+\epsilon} e^{-n/\sqrt{X}}.$$

If $nn_1 \cdots n_{k-1}$ is an odd square then Lemma 3.1 and partial summation gives that the sum over d in (3.3) is
$$\frac{X}{16\zeta(2)} \prod_{p | 2nn_1 \cdots n_{k-1}} \left(\frac{p}{p+1} \right) \int_1^2 W \left(\frac{n\sqrt{2\pi}}{\sqrt{Xt}} \right) dt + O(X^{\frac{1}{2}+\epsilon} e^{-n/\sqrt{X}}).$$

We use these two observations in (3.3). Note that the error terms contribute to (3.3) an amount $\ll X^{\frac{1}{2}+\epsilon} x^{(k-1)(\frac{3}{4}+\epsilon)} X^{\frac{3}{8}+\epsilon} \ll X^{\frac{39}{40}}$. It remains to estimate the main term contribution to (3.3). To analyze these terms let us write $n_1 \cdots n_{k-1}$ as rs^2 where r and s are odd and r is square-free. Then n must be of the form $r\ell^2$ where ℓ is odd. With this notation the main term contribution to (3.3) is
$$\frac{X}{8\zeta(2)} \sum_{\substack{rs^2 = n_1 \cdots n_{k-1} \\ n_1,\ldots,n_{k-1} \leq x}} \frac{1}{rs} \sum_{\ell \text{ odd}} \frac{1}{\ell} \int_1^2 \prod_{p | 2rs\ell} \left(\frac{p}{p+1} \right) W \left(\frac{r\ell^2 \sqrt{2\pi}}{\sqrt{Xt}} \right) dt.$$

Note that $r \leq x^{k-1} < X^{\frac{1}{10}}$, and an easy calculation gives that the sum over ℓ above is
$$= \prod_{p | 2rs} \left(\frac{p}{p+1} \right) \prod_{p \nmid 2rs} \left(1 - \frac{1}{p(p+1)} \right) \frac{1}{4} \log \frac{\sqrt{X}}{r} + O(1).$$

It follows that the main term contribution to (3.3) is
$$\gg X \log X \sum_{\substack{rs^2 = n_1 \cdots n_{k-1} \\ n_1,\ldots,n_{k-1} \leq x}} \frac{1}{rs} \prod_{p | 2rs} \left(\frac{p}{p+1} \right)$$
$$\gg X \log X \sum_{\substack{r \text{ odd and square-free} \\ s \text{ odd} \\ rs^2 \leq x}} \frac{d_{k-1}(rs^2)}{rs} \prod_{p | 2rs} \left(\frac{p}{p+1} \right) \gg X (\log X)(\log x)^{k-1+k(k-1)/2},$$

where the last bound follows by invoking Theorem 2 of [9]. We conclude that
$$S_1 \gg X (\log X)^{k(k+1)/2},$$

which when combined with (3.2) proves Theorem 2.

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